Journal of Engineering Mathematics 20 (1986) 73–79

An investigation of beam theory using the Airy stress function coupled with analytic function theory

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(Received August 8, 1985)

Summary

The Airy stress function, coupled with the theory of functions of a complex variable, is used to investigate the bending of plates in plane strain. In the limiting case of a thin plate, the theory reproduces the well-known result of elasticity theory known as Love's Equation [1] and systematic corrections in powers of the plate thickness are easily derived.

1. Introduction

The Airy stress function, coupled with the theory of analytic functions of a complex variable, is particularly useful for solving plane boundary-value problems in elasticity theory, because surface deformations can be directly related to applied surface forces by means of integral transforms [2]. The purpose of this paper is to apply this method of analysis to the case of a beam (or plate) under plane strain. For completeness, we give the equations of elasticity and of the theory of analytic functions, that are necessary to derive all of our results.

Consider the two-dimensional displacement field

$$\mathbf{d} = (d_1(x, y), d_2(x, y), 0). \tag{1.1}$$

The stress and strain tensors can be written as

$$U_{ij} = \frac{1}{2}\left(\frac{\partial d_i}{\partial x_j} + \frac{\partial d_j}{\partial x_i}\right), \tag{1.2}$$

$$\sigma_{ij} = \lambda\delta_{ij}(U_{ii}) + 2\mu U_{ij}. \tag{1.3}$$

The equilibrium equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{1.4}$$

which are automatically satisfied by the use of the Airy stress function $A(x, y)$,

$$\sigma_{xx} = \frac{\partial^2 A(x, y)}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 A(x, y)}{\partial x\, \partial y}, \quad \sigma_{yy} = \frac{\partial^2 A(x, y)}{\partial x^2}, \tag{1.5}$$

from which (1.3) yields

$$\begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix} = \begin{pmatrix} \frac{(\lambda + 2\mu)}{4\mu(\lambda+\mu)} \nabla^2 A(x, y) - \frac{1}{2\mu}\frac{\partial^2 A(x, y)}{\partial x^2}, & -\frac{1}{2\mu}\frac{\partial^2 A(x, y)}{\partial x\, \partial y} \\ -\frac{1}{2\mu}\frac{\partial^2 A(x, y)}{\partial x\, \partial y}, & \frac{(\lambda + 2\mu)}{4\mu(\lambda+\mu)} \nabla^2 A(x, y) - \frac{1}{2\mu}\frac{\partial^2 A(x, y)}{\partial y^2} \end{pmatrix} \tag{1.6}$$

The compatability equations, relating (1.2) and (1.6), then give the biharmonic equation

$$\nabla^4 A(x, y) = 0. \tag{1.7}$$

Taking the complex form of ∇^2,

$$\nabla^2 = 4\frac{\partial}{\partial z}\frac{\partial}{\partial z^*},$$

solutions of Equation (1.7) can be written as

$$A(x, y) = \mathrm{Re}\left[f_1(z) + \left(\frac{z + z^*}{2}\right) f_2(z)\right] \tag{1.8}$$

where

$$f_n(z) = U_n(x, y) + \mathrm{i}V_n(x, y), \quad n = 1, 2, \tag{1.9}$$

is an analytic function of $z = x + \mathrm{i}y$. From (1.8),

$$A(x, y) = U_1(x, y) + xU_2(x, y). \tag{1.10}$$

The displacements up to an irrelevant rotation can be obtained by integrating (1.6) to give

$$d_1(x, y) = \frac{(\lambda + 2\mu)}{2\mu(\lambda+\mu)} U_2(x, y) - \frac{1}{2\mu}\frac{\partial A(x, y)}{\partial x},$$

$$d_2(x, y) = \frac{(\lambda + 2\mu)}{2\mu(\lambda+\mu)} V_2(x, y) - \frac{1}{2\mu}\frac{\partial A(x, y)}{\partial y}. \tag{1.11}$$

The equations of analytic function theory that we need are the Cauchy-Riemann equations,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}, \tag{1.12}$$

and Cauchy's integral representation,

$$f(z)=\frac{1}{\pi \mathrm{i}}\int_c \frac{f(z')\,\mathrm{d}z'}{z'-z} \tag{1.13}$$

where z and z' are both on the closed contour of integration, c.

2. Elasticity equations

Here, we apply the elasticity theory, discussed in the introduction, to the case of an infinite plate of thickness l. Consider a section of the infinite plate under plane strain, to be represented by a strip in the complex plane, as shown in Fig. 1. Surface I is free and surface II is subjected to pressure and shear distributions $P(y)$ and $S(y)$, respectively. Such a situation is encountered, for example, when a plate is rigidly attached to a substrate, which is itself subject to plane deformation (see for example [3]). Zero external forces on I can, through equations (1.5), be simulated by assuming that both A and its normal derivative are zero on this surface, i.e.,

$$\begin{aligned} &A(-\tfrac{1}{2}l,\ y)=0,\\ &A_x(-\tfrac{1}{2}l,\ y)=\frac{\partial A}{\partial x}(-\tfrac{1}{2}l,\ y)=0. \end{aligned} \tag{2.1}$$

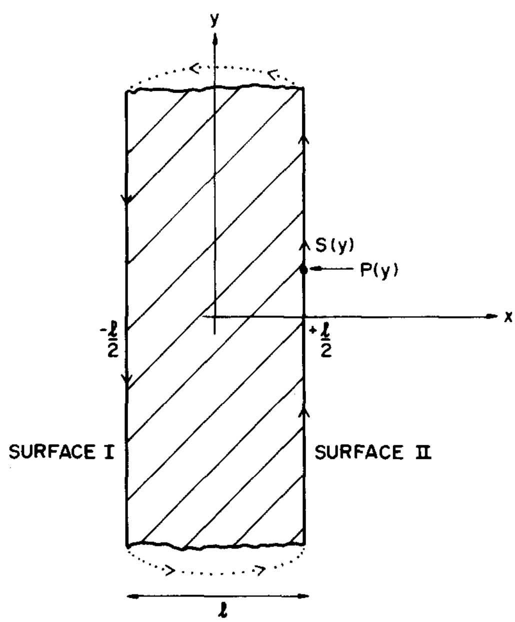

Figure 1. A diagrammatic representation of a plate of infinite length and width l, as an infinite strip of the complex plane, showing the closed counter used in the integration along the surface boundary.

The external stress on surface II implies

$$A(\tfrac{1}{2}l,\ y)=\int_{-\infty}^{y} P(y')(y-y')\,\mathrm{d}y' = A(y),$$

$$A_x(\tfrac{1}{2}l,\ y)=-\int_{-\infty}^{y} S(y')\,\mathrm{d}y' \qquad = A_x(y). \tag{2.2}$$

In order to simplify the notation, it is convenient to define

$$U_n^+(y)=U_n(\tfrac{1}{2}l,\ y),\quad U_n^-(y)=U_n(-\tfrac{1}{2}l,\ y), \tag{2.3}$$

$$U_n^s(y)=U_n^+(y)+U_n^-(y),\quad U_n^D(y)=U_n^+(y)-U_n^-(y),$$

and similar results for the functions V_n, where the subscript $n=1, 2$. Equations (1.10), (1.12), (2.1) and (2.2) give

$$A(\pm\tfrac{1}{2}l,\ y)=U_1^{\pm}(y)\pm\tfrac{1}{2}lU_2^{\pm}(y)$$

and

$$A_x(\pm\tfrac{1}{2}l,\ y)=\frac{\partial V_1^{\pm}(y)}{\partial y}\pm\tfrac{1}{2}l\frac{\partial V_2^{\pm}(y)}{\partial y}+U_2^{\pm}(y). \tag{2.4}$$

Or, in terms of sums and differences, we obtain our principal elasticity equations:

$$U_1^s(y)+\tfrac{1}{2}lU_2^D(y)=A(y),$$

$$U_1^D(y)+\tfrac{1}{2}lU_2^s(y)=A(y),$$

$$\frac{\partial V_1^s(y)}{\partial y}+U_2^s(y)+\tfrac{1}{2}l\frac{\partial V_2^D(y)}{\partial y}=A_x(y),$$

$$\frac{\partial V_1^D(y)}{\partial y}+U_2^D(y)+\tfrac{1}{2}l\frac{\partial V_2^s(y)}{\partial y}=A_x(y).$$

3. Analyticity equations

Cauchy's integral representation along the boundary shown in Fig. 1, with $z=-\tfrac{1}{2}l+\mathrm{i}y$, gives

$$U_n^-(y)+\mathrm{i}V_n^-(y)=-\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{U_n^-(y')+\mathrm{i}V_n^-(y')}{y'-y}\,\mathrm{d}y'$$

$$+\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{U_n^+(y')+\mathrm{i}V_n^+(y')}{y'-y-\mathrm{i}l}\,\mathrm{d}y', \tag{3.1}$$

and a similar expression for $z=\frac{1}{2}l+\mathrm{i}y$. By equating the real and imaginary parts and rearranging, one obtains the following set of equations:

$$U_n^s=\hat{L}V_n^D+\hat{M}V_n^D+\hat{N}U_n^s,\qquad U_n^D=\hat{L}V_n^s-\hat{M}V_n^s-\hat{N}U_n^D,$$
$$V_n^s=-\hat{L}U_n^D-\hat{M}U_n^D+\hat{N}V_n^s,\quad V_n^D=-\hat{L}U_n^s+\hat{M}U_n^s-\hat{N}V_n^D, \tag{3.2}$$

where $\hat{L}$, $\hat{M}$ and $\hat{N}$ are integral operators such that

$$\hat{L}U=\int_{-\infty}^{\infty}L(y'-y)U(y')\,\mathrm{d}y',$$

for example, with

$$L=\frac{1}{\pi}\frac{1}{y'-y},\quad M=\frac{1}{\pi}\frac{(y'-y)}{(y'-y)^2+l^2}\quad\text{and}\quad N=\frac{l}{\pi}\frac{1}{(y'-y)^2+l^2}.$$

Since these are convolutions, equations (3.2) become simple algebraic equations when expressed in terms of Fourier transforms, for example,

$$\tilde{U}_n^s(k)=\tilde{L}(k)\tilde{V}_n^D(k)+\tilde{M}(k)\tilde{V}_n^D(k)+\tilde{N}(k)\tilde{U}_n^s(k),\quad\text{etc.,} \tag{3.3}$$

where

$$\tilde{L}(k)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-\mathrm{i}k(y'-y))}{y'-y}\mathrm{d}y=-\mathrm{i}\epsilon(k), \tag{3.4}$$

$$\tilde{M}(k)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(y'-y)\exp(-\mathrm{i}k(y'-y))}{(y'-y)^2+l^2}\mathrm{d}y=-\mathrm{i}\epsilon(k)\exp(-|k|l), \tag{3.5}$$

$$\tilde{N}(k)=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-\mathrm{i}k(y'-y))}{(y'-y)^2+l^2}\mathrm{d}y=\exp(-|k|l) \tag{3.6}$$

and

$$\epsilon(k)=\begin{cases}-1, & k<0,\\ +1, & k>0.\end{cases}$$

Out of the four transformation equations obtained from analyticity, only 2 are independent; for example,

$$\tilde{U}_n^s(k)=-\mathrm{i}\epsilon(k)\tilde{V}_n^D(k)\frac{(1+\exp(-|k|l))}{(1-\exp(-|k|l))},$$

$$\tilde{U}_n^D=-\mathrm{i}\epsilon(k)\tilde{V}_n^s(k)\frac{(1-\exp(-|k|l))}{(1+\exp(-|k|l))}.$$

Thus, writing out these expressions explicitly for both f_1 and f_2 (i.e. $n=1,2$) and transforming the 4 elasticity equations, one obtains in total eight linear algebraic equations in eight unknowns:

$$\tilde{U}_1^s(k) = -i\epsilon(k)\tilde{V}_1^D(k)\frac{(1+\exp(-|k|l))}{(1-\exp(-|k|l))},$$

$$\begin{aligned}
\tilde{U}_2^s(k) &= -i\epsilon(k)\tilde{V}_2^D(k)\frac{(1+\exp(-|k|l))}{(1-\exp(-|k|l))},\\
\tilde{U}_1^D(k) &= -i\epsilon(k)\tilde{V}_1^s(k)\frac{(1-\exp(-|k|l))}{(1+\exp(-|k|l))},\\
\tilde{U}_2^D(k) &= -i\epsilon(k)\tilde{V}_2^s(k)\frac{(1-\exp(-|k|l))}{(1+\exp(-|k|l))},\\
\tilde{U}_1^s(k) &+ \tfrac{1}{2}l\tilde{U}_2^D(k) = \tilde{A}(k),\\
\tilde{U}_1^D(k) &+ \tfrac{1}{2}l\tilde{U}_2^s(k) = \tilde{A}(k),\\
-ik\tilde{V}_1^s(k) &+ \tilde{U}_2^s(k) - \tfrac{1}{2}ikl\tilde{V}_2^D(k) = \tilde{A}_x(k),\\
-ik\tilde{V}_1^D(k) &+ \tilde{U}_2^D(k) - \tfrac{1}{2}ikl\tilde{V}_2^s(k) = \tilde{A}_x(k),
\end{aligned} \tag{3.7}$$

Solving these equations, one can show for example, that

$$\tilde{U}_2^s(k) = \tilde{O}(k)\left[-\frac{12}{l^3k^2}\tilde{A}(k)\right] + \tilde{P}(k)\left[\frac{6}{l^2k^2}\tilde{A}_x(k)\right] \tag{3.8}$$

where

$$\tilde{O}(k) = \frac{|k|^3l^3(1+\exp(-l|k|))^2}{12[1-\exp(-2l|k|)-2l|k|\exp(-l|k|)]} \tag{3.9}$$

and

$$\tilde{P}(k) = \frac{l^2k^2(1-\exp(-2l|k|)}{6[1-\exp(-2l|k|)-2l|k|\exp(-l|k|))]}. \tag{3.10}$$

These equations and those that can be similarly derived for all other surface displacements, are a complete solution to this elasticity problem for arbitrary plate thickness l and arbitrary external stress. Expressed in co-ordinate space, they give surface displacements directly as integral transforms of the applied surface stress.

For small l,

$$\tilde{O}(k) = \left[1 + \tfrac{1}{5}l^2k^2\right], \quad \tilde{P}(k) = \left[1 + \tfrac{7}{60}l^2k^2\right], \tag{3.11}$$

and equations (1.11), (3.8) and (3.11) give for the mean-surface displacements

$$\frac{d^2 d_1}{dy^2} = \frac{(\lambda + 2\mu)}{\mu(\lambda + \mu)}\left[\frac{3A(y)}{l^3} - \frac{3A_x(y)}{2l^2} - \frac{3}{5l}P(y)\right] - \frac{(3\lambda - 4\mu)}{40\mu(\lambda + \mu)}\frac{dS(y)}{dy}. \quad (3.12)$$

where the moments $A(y)$ and $A_x(y)$ are related to the applied stresses $P(y)$ and $S(y)$ by equations (2.2). For small l, the first term on the right-hand side of equation (3.12) dominates and is equivalent to Love's result for the bending of thin plates [1].

4. Conclusions

This work has shown the power and utility of the Airy stress function, coupled with the theory of functions of a complex variable, in a two-dimensional formulation of beam theory. In this formulation, appropriate displacements can be isolated and expressed directly in terms of integral transforms on the applied stresses. The information concerning the displacements of the system is therefore retained until the end of the calculation in the form of 8 algebraic equations. This is particularly advantageous over the methods normally employed in this type of problem which necessitate successive approximations being made throughout the calculation.

In the limiting case of a thin plate, the theory reproduces the well-known result of elasticity theory known as Love's Equation, for which systematic corrections in powers of the plate thickness can be easily obtained.

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